Pre-class Warm-up!!!

What is $\dfrac{\partial^2 f}{\partial x \partial z}$ when $f(x,y,z) = xy + z\sin(x)$?

✓ a. $\cos(x)$

b. $y + \sin(x)$

c. $z\cos(x)$

d. y

e. None of the above

Note $\dfrac{\partial^2 f}{\partial x \partial z}$ means $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial z}\right)$

and $\dfrac{\partial^2 f}{\partial z \partial x} = \dfrac{\partial}{\partial z}\left(y + z\cos x\right)$

$= \cos x \quad = \dfrac{\partial^2 f}{\partial x \partial z}$

3.1 Iterated partial derivatives

We have already been using these in showing that curl(grad(f)) = (0, 0, 0) and div(curl(F)) = 0.

In Section 3.1 they define them and prove symmetry of the mixed partial derivatives using the mean value theorem.

HW questions are all: calculate these mixed partial derivatives, verify that this function satisfies this partial differential equation.

3.2 Taylor's theorem.

Question. What is the Taylor expansion of $f(x) = x^2$ about $x = 1$?

a. $f(x) = 1 - 2(x+1) + (x+1)^2$

✓ b. $f(x) = 1 + 2(x-1) + (x-1)^2$

c. $f(x) = 1 - 2(1-x) + (1-x)^2$

✓ d. $f(1+x) = 1 + 2x + x^2$

e. None of the above.

Taylor expansion of $g(x)$ about 0 is a series $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

About $x = 1$ it is
$g(1+h) = a_0 + a_1 h + a_2 h^2 + \cdots$ (different a_i)
or $g(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + \cdots$

We learn:

- What Taylor polynomials are.
- What a Taylor series is.
- What Taylor approximations are.
- The form of the terms in a Taylor polynomial.
- Taylor's theorem
- How to write the degree 1 and 2 polynomials in terms of the gradient and the Hessian matrix.

We don't need to know

- The forms of the remainder terms

= discrepancy between the Taylor polynomial and the true.

The form of the coefficients in the Taylor series (1-variable case)

Do this first for the expansion about 0:

f(x) = a_0 + a_1 x + a_2 x^2 +

$f(0) = a_0 + a_1 0 + a_2 0 \cdots = a_0$

Apply $\frac{d}{dx}$ to both sides:

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$f'(0) = a_1$

Apply $\frac{d}{dx}$ to both sides

$f^{(2)}(x) = 2a_2 + 3\cdot 2\cdot a_3 x + \cdots$

$f^{(2)}(0) = 2a_2 \qquad a_2 = \frac{1}{2} f^{(2)}(0)$

$a_n = \frac{1}{n!} f^{(n)}(0)$

Next: the expansion about c :

f(c+h) = a_0 + a_1 h + a_2 h^2 +

Do the same, but evaluate when $h=0$, so evaluate at c.

$f(c) = a_0$

Apply $\frac{d}{dx}$ $f'(c+h) = a_1 + 2a_2 h \cdots$

$f'(c) = a_1$

$\vdots$

$a_n = \frac{1}{n!} f^{(n)}(c)$.

The expansion about c :

f(c+h) = a_0 + a_1 h + a_2 h^2 + ...

= a_0 + a_1 h + ... + a_2 h^n + R_n(c,h)

= Taylor polynomial of degree n
+ Remainder term of degree n

where a_i = $\frac{1}{i!} f^{(i)}(c)$

Taylor's theorem:
R_n(c,h) / h^n -> 0 as h -> 0

When $n = 1$ the Taylor polynomial is

$$a_0 + a_1 h = f(c) + f'(c)h$$

which approximates $f(c+h)$ linearly:

$$f(c+h) \approx f(c) + f'(c)\,h$$

The remainder term is

$$R_1(c,h) = f(c+h) - f(c) - f'(c)h$$

and

$$\frac{R_1(c,h)}{h} = \frac{f(c+h) - f(c) - f'(c)h}{h}$$

$\rightarrow 0$ as $h \rightarrow 0$.

Taylor series with more than one variable

Now $f: R^n \to R$
and $c = (c_1,\dots,c_n)$, $h = (h_1,\dots,h_n)$ lie in R^n

linear terms

$$f(c+h) = a_{00\dots0} + \underbrace{a_{10\dots0}h_1 + a_{010\dots0}h_2 + \dots + a_{0\dots1}h_n}$$

$$+ a_{20\dots0}h_1^2 + a_{020\dots0}h_2^2 + a_{110\dots0}h_1h_2 + \dots$$

Subscripts record the degrees of the $h_1, \dots, h_n$

What are the coefficients a ? Do the same as 1-variable:

Put $\underline{h} = \underline{0}$: $f(\underline{c}) = a_{0\dots0}$

Apply $\frac{\partial}{\partial x_1} f = a_{10\dots0} + 2a_{20\dots0}h_1 + \dots$

Put $\underline{h} = 0$. $a_{1000} = \frac{\partial f}{\partial x_1}(\underline{c})$

$$a_{20} = \frac{1}{2!}\frac{\partial^2}{\partial x_1} f(c) \qquad a_{11} = \frac{\partial}{\partial x_1 \partial x_2} f(c)$$

Example: $n = 2$ $f(x,y)$

degree 1

$$f(x+h_1, y+h_2) = a_{00} + a_{10}h_1 + a_{01}h_2$$

$$+ a_{20}h_1^2 + a_{11}h_1h_2 + a_{02}h_2^2 \quad \text{degree 2}$$

$$+ a_{30}h_1^3 + a_{21}h_1^2h_2 + a_{12}h_1h_2^2 + a_{03}h_2^3$$

degree 3

$+ \dots$

$$a_{i_1 i_2 \dots i_t} h_1^{i_1} h_2^{i_2} \dots h_t^{i_t}$$

We see the best linear approximation and the best quadratic approximation to f around c.

The Hessian matrix H

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_3} & & & \\ & & & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

entries : $\frac{\partial^2 f}{\partial x_i \partial x_j}$ $\quad 1 \le i, j \le n$.

evaluated at $\underline{c}$.

Taylor poly of degree 2 is

$$f(\underline{c} + \underline{h}) \approx f(\underline{c}) + Df(\underline{c})(\underline{h}) + \frac{1}{2} \underline{h}^T H \underline{h}$$

Example: Find the first and second degree Taylor polynomials and the Hessian matrix for f(x,y) = sin(xy) at c = (1, π/2). Use these to approximate f(1.1,π/2).